Sec. 8.3 Trigonometric Functions: Relationships and Graphs

Inverse Trigonometric Functions and Their Properties (Identities):

$$\text{cosecant}\theta = \csc\theta = \frac{1}{\sin\theta} \qquad \text{secant}\theta = \sec\theta = \frac{1}{\cos\theta} \qquad \text{cotangent}\theta = \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Negative Identities (Even/Odd Properties):

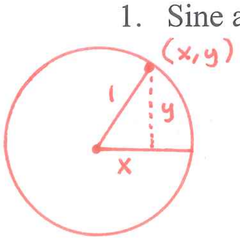
$$\sin(-\theta) = -\sin\theta \qquad \tan(-\theta) = -\tan\theta \qquad \cos(-\theta) = \cos\theta$$

Also works with their inverses:

$\csc(-\theta) = -\csc\theta$ $\qquad$ $\cot(-\theta) = -\cot\theta$ $\qquad$ $\sec(-\theta) = \sec\theta$

Pythagorean Identities: (Create and remember first and you can derive the other two.)

1. Sine and Cosine:



$x^2 + y^2 = 1^2$

$\sin^2\theta + \cos^2\theta = 1$

2. Divide each term by $\sin^2\theta$ to find the second:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

3. Divide each term by $\cos^2\theta$ to find the third:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Theorem: Let t be a real number and let P = (x,y) be a point on the unit circle that corresponds to angle t.

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y} \qquad \sec\theta = \frac{1}{x} \qquad \csc\theta = \frac{1}{y}$$

Remember that all points in the unit circle are written as $(\cos\theta, \sin\theta)$.

Ex: Let t be a real number and P = $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ be the point on the unit circle that corresponds to t.

Find the six trigonometric values.

$\boxed{\sin t = \frac{\sqrt{3}}{2}}$

$\boxed{\cos t = -\frac{1}{2}}$

$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ $\boxed{\csc t = \frac{2\sqrt{3}}{3}}$

$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -\frac{2}{1}$ $\boxed{\sec t = -2}$

$\tan t = \frac{\sin t}{\cos t} = \frac{\sqrt{3}}{2} \div -\frac{1}{2} = \frac{\sqrt{3}}{2} \cdot -\frac{2}{1}$ $\boxed{\tan t = -\sqrt{3}}$

$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ $\boxed{\cot t = -\frac{\sqrt{3}}{3}}$

Ex: Find the exact values of the trig functions of:

a. 0° $(1,0)$

$\sin 0° = \boxed{0}$ $\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = \boxed{1}$

$\cos 0° = \boxed{1}$

$\tan 0° = \frac{0}{1} = \boxed{0}$ $\cot 0° = \frac{1}{0}$

$\csc 0° = \frac{1}{\sin 0°} = \frac{1}{0} = \boxed{\text{undefined}}$

$\csc 0° = \boxed{\text{undefined}}$

b. 90° $(0,1)$

$\sin 90° = \boxed{1}$ $\cos 90° = \boxed{0}$

$\csc 90° = \frac{1}{1} = \boxed{1}$

$\sec 90° = \frac{1}{0} = \boxed{\text{undefined}}$

$\tan 90° = \frac{1}{0} = \boxed{\text{undefined}}$

$\cot 90° = \frac{0}{1} = \boxed{0}$

c. 180° $(-1,0)$

$\sin 180° = \boxed{0}$ $\cos 180° = \boxed{-1}$

$\csc 180° = \frac{1}{0} = \boxed{\text{undefined}}$

$\sec 180° = \frac{1}{-1} = \boxed{-1}$

$\tan 180° = \frac{0}{-1} = \boxed{0}$

$\cot 180° = \frac{-1}{0}$

$= \boxed{\text{undefined}}$

d. 270° $(0,-1)$

$\sin 270° = \boxed{-1}$ $\cos 270 = \boxed{0}$

$\csc 270° = \frac{1}{-1} = \boxed{-1}$

$\sec 270° = \frac{1}{0} = \boxed{\text{undefined}}$

$\tan 270° = -\frac{1}{0} = \boxed{\text{undefined}}$

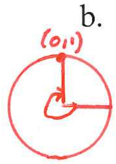
$\cot 270° = \frac{0}{-1} = \boxed{0}$

Ex: Find the exact value of the following:

a. sin (3π)

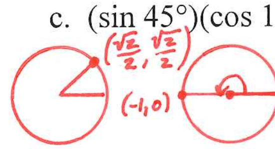
$\boxed{0}$

b. cos (-270°)



$\boxed{0}$

c. (sin 45°)(cos 180°)



$\frac{\sqrt{2}}{2} \cdot -1 = \boxed{-\frac{\sqrt{2}}{2}}$

d. tan π/4 – sin 3π/2

$\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} - 1$

$1 - (-1)$

$\boxed{2}$

e. $(\sec \pi/4)^2$ + csc π/2

$\left(\frac{1}{\cos\frac{\pi}{4}}\right)^2 + \frac{1}{\sin\frac{\pi}{2}}$

$\left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2 + \frac{1}{1}$

$\left(\frac{2}{\sqrt{2}}\right)^2 + 1$

$\frac{4}{2} + 1 = \boxed{3}$

f. tan 315°

$\frac{\sin 315°}{\cos 315°}$

$-\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2}$

$\boxed{-1}$

**Now try these out on your calculator. Do you get the same answer?? Remember your mode!

Ex: Find the exact values of the trigonometric functions for π/6 = 30°. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\sin 30° = \boxed{\frac{1}{2}}$

$\cos 30° = \boxed{\frac{\sqrt{3}}{2}}$

$\csc 30° = \boxed{2}$

$\sec 30° = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \boxed{\frac{2\sqrt{3}}{3}}$

$\tan 30° = \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \boxed{\frac{\sqrt{3}}{3}}$

$\cot 30° = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{3\sqrt{3}}{3}$

$= \boxed{\sqrt{3}}$

Periodic Properties:

$\sin(\theta+2\pi k)=\sin\theta$	$\cos(\theta+2\pi k)=\cos\theta$	$\tan(\theta+\pi k)=\tan\theta$
$\csc(\theta+2\pi k)=\csc\theta$	$\sec(\theta+2\pi k)=\sec\theta$	$\cot(\theta+\pi k)=\cot\theta$

Ex. Find the exact values using periodic properties of:

a. $\sin\left(\frac{17\pi}{4}\right)$ $=\sin\left(4\frac{1}{4}\pi\right)$

$=\sin\left(\frac{\pi}{4}\right)$

$=\boxed{\frac{\sqrt{2}}{2}}$

b. $\tan\left(\frac{5\pi}{4}\right)$

$=\tan\left(\pi+\frac{\pi}{4}\right)$

$=\tan\left(\frac{\pi}{4}\right)$

$=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=\boxed{1}$

c. $\cos(5\pi)$

$\cos(4\pi+\pi)$

$\cos\pi=\boxed{-1}$

Ex: Given $\sin\theta=\frac{\sqrt{5}}{5}$ and $\cos\theta=\frac{2\sqrt{5}}{5}$, find the exact values of the four remaining trigonometric functions using identities.

$\csc\theta=\frac{1}{\sin\theta}$

$=\frac{1}{\frac{\sqrt{5}}{5}}$

$=1\cdot\frac{5}{\sqrt{5}}$

$=\frac{5}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$=\frac{5\sqrt{5}}{5}=\boxed{\sqrt{5}}$

$\sec\theta=\frac{1}{\cos\theta}$

$=\frac{5}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$=\frac{5\sqrt{5}}{2\cdot 10}$

$=\boxed{\frac{\sqrt{5}}{4}}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}$

$=\frac{\sqrt{5}}{5}\div\frac{2\sqrt{5}}{5}$

$=\frac{\sqrt{5}}{5}\cdot\frac{5}{2\sqrt{5}}$

$=\boxed{\frac{1}{2}}$

$\cot\theta=\frac{1}{\tan\theta}$

$=\frac{1}{\frac{1}{2}}$

$=1\cdot\frac{2}{1}$

$=\boxed{2}$

Ex: Find the exact value of each expression without a calculator.

a. $\tan 20°-\frac{\sin 20°}{\cos 20°}$

$\tan 20°-\tan 20°$

0

b. $\sin^2\left(\frac{\pi}{12}\right)+\frac{1}{\sec^2\left(\frac{\pi}{12}\right)}$

$\sin^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{\pi}{12}\right)=\boxed{1}$

Ex: Given that the sin θ = 1/3 and cos θ < 0, find the exact values of each of the remaining trigonometric functions.

$\sin^2\theta+\cos^2\theta=1$

$\left(\frac{1}{3}\right)^2+\cos^2\theta=1$

$\frac{1}{9}+\cos^2\theta=1$

$\cos^2\theta=\frac{8}{9}$

$\cos\theta=\pm\frac{\sqrt{8}}{3}$

$\cos\theta=\boxed{-\frac{\sqrt{8}}{3}}$

$\sec\theta=-\frac{3}{\sqrt{8}}\cdot\frac{\sqrt{8}}{\sqrt{8}}=\boxed{-\frac{3\sqrt{8}}{8}}$

$\csc\theta=\frac{1}{\sin\theta}$

$=1\div\frac{1}{3}$

$=1\cdot 3$

$=\boxed{3}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}$

$=\frac{1}{3}\div-\frac{\sqrt{8}}{3}$

$=\frac{1}{3}\cdot-\frac{3}{\sqrt{8}}$

Use this for $\cot\theta$ → $=-\frac{1}{\sqrt{8}}\cdot\frac{\sqrt{8}}{\sqrt{8}}$

$=\boxed{-\frac{\sqrt{8}}{8}}$

$\cot\theta=\frac{1}{\tan\theta}$

$=\frac{-\sqrt{8}}{1}$

$=\boxed{-\sqrt{8}}$

Finding the Values of the Trig Functions when One is Known
Method 1:

a. Draw a circle showing the location of the angle θ and the point P = (x, y) that corresponds to θ. The radius of the circle is $r = \sqrt{x^2 + y^2}$.
b. Assign a value to two of the three variables x, y, r based on the value of the given trig function.
c. Use the fact that P lies on the circle $x^2 + y^2 = r^2$ to find the value of the missing variable.
d. Apply any necessary theorems.

Method 2:

Use appropriately selected identities to find the value of each of the remaining trig functions.

→ $\cos\theta < 0$ if $\tan\theta > 0$

Ex: Given that tan θ = 1/2 and sin θ < 0, find the exact value of each of the remaining five trig functions of θ.

$\tan^2\theta + 1 = \sec^2\theta$

$\left(\frac{1}{2}\right)^2 + 1 = \sec^2\theta$

$\frac{1}{4} + 1 = \sec^2\theta$

$\frac{5}{4} = \sec^2\theta$

$\pm\frac{\sqrt{5}}{2} = \sec\theta$

Must be neg (cos is < 0) $\boxed{-\frac{\sqrt{5}}{2} = \sec\theta}$

$\cos\theta = -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

$\boxed{\cos\theta = -\frac{2\sqrt{5}}{5}}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{1}{2} = \frac{\sin\theta}{-\frac{2\sqrt{5}}{5}}$

$\frac{1}{2} \cdot -\frac{2\sqrt{5}}{5} = \sin\theta$

$\boxed{-\frac{\sqrt{5}}{5} = \sin\theta}$

$\csc\theta = -\frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

$= -\frac{5\sqrt{5}}{5}$

$\boxed{\csc\theta = -\sqrt{5}}$

$\cot\theta = \frac{1}{\tan\theta}$

$= \frac{2}{1}$

$\boxed{\cot\theta = 2}$

HW: pg 339- 340 #2-16 evens, 22, 24, 26, 28, 32, 37 (Rationalize whenever possible.)